ON EXPLOSIONS ABOVE SURFACES OF LIQUIDS

(K ZADACHE O VZRYVE NAD POVERKHNOST'IU ZHIDKOSTI)

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The problem of explosions above surfaces of liquids is studied under the following assumptions:

(a) the effect of the external explosion on the motion of the free surface of the liquid can be simulated by an appropriate unsteady pressure distribution over a time-varying area;

(b) the motion of the liquid is linearized, which can be justified by the large difference between densities of liquid and gas;

(c) the liquid is considered as incompressible. This assumption increases in validity as the ratio of the speed of sound in the real liquid to the propagation speed of the shockwave over the surface increases.

These assumptions make it possible to reduce the problem at hand to a problem of infinitely small surface wave disturbances over a heavy incompressible ideal liquid. Apparently, this conceptual setting was first used by Lamb [1] in connection with problems of long surface waves. In modern times, wave motion from this point of view has been studied in some detail by Finkelstein [2]. A similar approach was also used by Voit [3], Cherkesov [4], and others.

From the results for explosions above heavy liquids, it is easy to derive the motion of free surfaces of weightless fluids by letting the gravitational constant g approach zero. This condition corresponds to the initial effects of the explosion when the pressure forces dominate the gravitational forces.

1. General expressions for the potential. Let a given pressure $p_0(x, y, t)$ be applied to free surface of a liquid extending to a depth h. The velocity potential in the liquid $\phi(x, y, z, t)$, satisfies Laplace's equation, $\Delta\phi = 0$; the system of coordinates is chosen as usual in the theory of surface waves* and t is time. If $\zeta(x, y, t)$ represents the displacement of the free surface from the plane $z = 0$, we can derive the boundary conditions at this surface from the Cauchy-Lagrange integral, utilizing the basic assumptions of the theory of infinitesimal waves:

$$\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial z} = -\frac{1}{\rho}\frac{\partial p_0(x, y, t)}{\partial t} \qquad \text{for } z = 0 \tag{1.1}$$

One step of this derivation leads to a relation from which we may find the displacement $\zeta(x, y, t)$.

$$\left.\frac{\partial\varphi}{\partial t}\right|_{z=0} + g\zeta(x, y, t) = -\frac{p_0(x, y, t)}{\rho} \tag{1.2}$$

At the bottom we obviously have

$$\frac{\partial\varphi}{\partial z} = 0 \qquad \text{for } z = -h \tag{1.3}$$

In dealing with an explosion over an initially calm surface we adopt as initial conditions:

$$\zeta = 0, \qquad \partial\zeta / \partial t = 0 \qquad \text{for } t = 0$$

By means of Equation (1.2), the initial conditions can be expressed as follows:

$$\left.\frac{\partial\varphi}{\partial t}\right|_{\substack{z=0\\ t=0}} = -\frac{p_0(x, y, 0)}{\rho}, \qquad \left.\frac{\partial^2\varphi}{\partial t^2}\right|_{\substack{z=0\\ t=0}} = -\frac{1}{\rho}\left.\frac{\partial p_0(x, y, t)}{\partial t}\right|_{t=0} \tag{1.4}$$

A two-dimensional Fourier representation is used for the desired potential function:

$$\varphi(x, y, z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi^*(\xi, \eta, z, t)\, e^{-i(\xi x + \eta y)}\, d\xi\, d\eta \tag{1.5}$$

where

$$\varphi^*(\xi, \eta, z, t) = -\left[p_0^*(\xi, \eta, 0)\sin\sigma t + \int_0^t \frac{\partial p_0^*(\xi, \eta, \tau)}{\partial\tau}\sin\sigma(t-\tau)\, d\tau\right]\frac{\operatorname{ch}\varkappa(z+h)}{\rho\sigma\operatorname{ch}\varkappa h}$$

$$\varkappa = \sqrt{\xi^2 + \eta^2}, \qquad \sigma = \sqrt{g\varkappa\, th\, \varkappa h}$$

and the asterisk denotes the Fourier transform.

* x, y axes in the undisturbed plane of the free surface; z axis upward, opposite to gravity.

2. Point-explosion above a body of liquid with finite depth. In this case the effect of the explosion on the fluid begins when the spherical shockwave touches the surface of the fluid. Subsequently, the effect is felt on the free surface in a region of imposed pressure variations. Evidently, this region is a circle, the radius of which is a known function of time, denoted by $r_0(t)$.

Assuming that the pressure is always finite, the axial symmetry of the problem leads to the expression for the velocity potential:

$$\varphi(r, z, t) = -\frac{1}{\rho}\int_0^\infty \xi \frac{\operatorname{ch}\xi(z+h)}{\operatorname{ch}\xi h} J_0(r\xi)\frac{\sin\sigma t}{\sigma}\, d\xi \int_0^{r_0(0)} a p_0(a, 0) J_0(a\xi)\, da -$$

$$-\frac{1}{\rho}\int_0^\infty \xi \frac{\operatorname{ch}\xi(z+h)}{\operatorname{ch}\xi h} J_0(r\xi)\, d\xi \int_0^t \frac{\sin\sigma(t-\tau)}{\sigma}\, d\tau \int_0^\infty a\frac{\partial p_0(a,\tau)}{\partial\tau} J_0(a\xi)\, da$$

$$\sigma = \sqrt{g\xi \operatorname{th}\xi h} \tag{2.1}$$

Here $J_q(u)$ is a Bessel function of order q, r is the radius in a cylindrical coordinate system, and $p_0(a, \tau)$ represents the disturbing pressure.

In the expression (2.1) the function $p_0(a, \tau)$ differs from zero only inside a circle of radius $a < r_0(\tau)$, and is identically zero outside. Hence

$$\int_0^\infty a\frac{\partial p_0(a,\tau)}{\partial\tau} J_0(a\xi)\, da = \frac{\partial}{\partial\tau}\int_0^{r_0(\tau)} a p_0(a,\tau) J_0(a\xi)\, da \tag{2.2}$$

so that Formula (2.1) is easily reduced to:

$$\varphi(r, z, t) = -\frac{1}{\rho}\int_0^\infty \xi \frac{\operatorname{ch}\xi(z+h)}{\operatorname{ch}\xi h} J_0(r\xi)\, d\xi \int_0^t \cos\sigma(t-\tau)\, d\tau \int_0^{r_0} a p_0(a,\tau) J_0(a\xi)\, da$$

Let us find the expression for the shape of the free surface of the liquid. Clearly, we must evaluate the limit:

$$\lim_{z\to 0}\frac{\partial\varphi}{\partial t} = -\frac{1}{\rho}\lim_{z\to 0}\int_0^\infty \xi\frac{\operatorname{ch}\xi(z+h)}{\operatorname{ch}\xi h} J_0(r\xi)\, d\xi \int_0^{r_0(t)} a p_0(a, t) J_0(a\xi)\, da +$$

$$+\frac{g}{\rho}\lim_{z\to 0}\int_0^\infty \xi^2 \operatorname{th}\xi h \frac{\operatorname{ch}\xi(z+h)}{\operatorname{ch}\xi h} J_0(r\xi)\, d\xi \int_0^t \frac{\sin\sigma(t-\tau)}{\sigma}\, d\tau \int_0^{r_0} a p_0(a,\tau) J_0(a\xi)\, da$$

Keeping in mind that the integral in the first part of this expression is equal to $p_0(r, t)$, we are led to the following form of the free surface:

$$\zeta(r, t) = -\frac{1}{\rho}\lim_{z\to 0}\int_0^\infty \xi^2 \operatorname{th}\xi h \frac{\operatorname{ch}\xi(z+h)}{\operatorname{ch}\xi h} J_0(r\xi)\, d\xi \int_0^t \frac{\sin\sigma(t-\tau)}{\sigma} d\tau \int_0^r a p_0(a, \tau) J_0(\xi)\, da$$

Let the function $p_0(a, \tau)$ be given as a series in even powers of the radius a on the surface:

$$p_0(a, \tau) = \sum_{n=0}^{\infty} \lambda_n(\tau) a^{2n} \qquad (2.3)$$

where $\lambda_n(\tau)$ are known functions of time. Using the identity

$$\int_0^{r_0} a^{2n+1} J_0(a\xi)\, da = n!\, r_0^{2n+2} \sum_{m=0}^{n} (-1)^m \frac{2^m}{(n-m)!} \frac{J_{m+1}(r_0\xi)}{(r_0\xi)^{m+1}}$$

we arrive at the series expression for the free surface:

$$\zeta(r, t) = -\frac{1}{\rho} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \alpha_{nm}\zeta_{nm}(r, t), \qquad \alpha_{nm} = \frac{(-1)^m 2^m n!}{(n-m)!} \qquad (2.4)$$

$$\zeta_{nm}(r, t) = \int_0^t \lambda_n(\tau) r_0^{2n-m+1}(\tau)\, d\tau \lim_{z\to 0}\int_0^\infty \xi \operatorname{th}\xi h \frac{\operatorname{ch}\xi(z+h)}{\operatorname{ch}\xi h} J_0(r\xi) \times$$
$$\times \frac{J_{m+1}(r_0\xi)}{\xi^m} \frac{\sin\sigma(t-\tau)}{\sigma} d\xi \qquad (2.5)$$

In the expressions for ζ_{nm}, the order of integration is inverted.

3. The shape of the free surface in the case of infinite depth. In this case, the limiting process $h \to \infty$ yields.

$$\zeta_{nm}(r,t) = \int_0^t \lambda_n(\tau) r_0^{2n-m+1}(\tau)\, d\tau \lim_{z\to 0}\int_0^\infty \xi e^{\xi z} J_0(r\xi) \frac{J_{m+1}(r_0\xi)}{\xi^m} \frac{\sin\sqrt{g\xi}(t-\tau)}{\sqrt{g\xi}} d\xi \qquad (3.1)$$

Let us seek the limiting value of the integral S_m as $z \to 0$:

$$S_m = \int_0^\infty \xi e^{\xi z} J_0(r\xi) \frac{J_{m+1}(r_0\xi)}{\xi^m} \frac{\sin\sqrt{g\xi}(t-\tau)}{\sqrt{g\xi}} d\xi \qquad (3.2)$$

Using Parseval's theorem for the Hankel transforms of first order [5]

$$S_m = \int_0^\infty \xi\chi(\xi)\psi(\xi)\, d\xi = \int_0^\infty u\chi^*(u)\psi^*(u)\, du$$

Here:

$$\chi(\xi) = \frac{J_{m+1}(r_0\xi)}{\xi^m}, \qquad \psi(\xi) = e^{\xi z} J_0(r\xi) \frac{\sin\sqrt{g\xi}\,(t-\tau)}{\sqrt{g\xi}}$$

$$\chi^*(u) = \int_0^\infty \frac{J_{m+1}(r_0\xi)\, J_1(u\xi)}{\xi^{m-1}} d\xi, \qquad \psi^*(u) = \int_0^\infty \xi e^{\xi z} J_0(r\xi)\, J_1(u\xi) \frac{\sin\sqrt{g\xi}\,(t-\tau)}{\sqrt{g\xi}} d\xi$$

We note that for $m = 0$ the conditions for the applicability of Parseval's theorem are not satisfied. Hence the preceding transforms are valid only for $m > 0$. For $m = 0$, the limiting value of the integral (3.2) is obtained differently, namely by analytic continuation from $m > 0$. We note that for $m > 0$, $\chi^*(u)$ represents a known Sonine integral [6]

$$\chi^*(u) = \frac{1}{r_0^{m+1}} \frac{u(r_0^2 - u^2)^{m-1}}{2^{m-1}(m-1)!} \qquad (0 < u < r_0) \tag{3.3}$$

In order to evaluate $\psi^*(u)$ at $z = 0$ we first note that

$$\psi^*(u) = -\frac{\partial}{\partial u} \int_0^\infty e^{\xi z} J_0(r\xi)\, J_0(u\xi) \frac{\sin\sqrt{g\xi}\,(t-\tau)}{\sqrt{g\xi}} d\xi$$

Let us seek the limiting value of the integral, which is to be differentiated with respect to u. We apply Parseval's theorem to the Hankel transform of zero order

$$L = \int_0^\infty e^{\xi z} J_0(r\xi)\, J_0(u\xi) \frac{\sin\sqrt{g\xi}\,(t-\tau)}{\sqrt{g\xi}} d\xi = \int_0^\infty \xi e^{\xi z} \frac{J_0(r\xi)\, J_0(u\xi)}{\xi^\delta} \frac{\sin\sqrt{g\xi}\,(t-\tau)}{\xi^{1-\delta}\sqrt{g\xi}} d\xi$$

where δ is a parameter between zero and unity which will be made to approach zero. We have

$$L = \int_0^\infty \xi\chi_1(\xi)\,\psi_1(\xi)\, d\xi = \int_0^\infty w\chi_1^*(w)\,\psi_1^*(w)\, dw \tag{3.4}$$

Here

$$\chi_1(\xi) = \frac{J_0(r\xi_0)\, J_0(u\xi)}{\xi^\delta}, \quad \chi_1^*(w) = \int_0^\infty J_0(r\xi)\, J_0(u\xi)\, J_0(w\xi)\, \xi^{1-\delta} d\xi$$

$$\psi_1(\varepsilon) = \frac{\sin\sqrt{g\xi}\,(t-\tau)}{\xi^{1-\delta}\sqrt{g\xi}} e^{\xi z}, \quad \psi_1^*(w) = \int_0^\infty e^{\xi z} J_0(w\xi) \frac{\sin\sqrt{g\xi}\,(t-\tau)}{\sqrt{g\xi}} \xi^\delta d\xi$$

The conditions of applicability of Parseval's theorem are fulfilled only for $\delta > 0$. However, inasmuch as the left-hand side of (3.4) represents a continuous function L, which is independent of δ, we deduce the following equality by letting δ approach zero:

$$L = \lim_{\delta\to 0} \int_0^\infty w\chi_1^*(w)\psi_1^*(w)\,dw = \int_0^\infty w\chi_{10}^*(w)\,\psi_{10}^*(w)\,dw$$

Here

$$\chi_{10}^*(w) = \int_0^\infty J_0(r\xi)\,J_0(u\xi)\,J_0(w\xi)\,\xi\,d\xi,\quad \psi_{10}^*(w) = \int_0^\infty e^{\xi z} J_0(w\xi)\frac{\sin\sqrt{g\xi}\,(t-\tau)}{\sqrt{g\xi}}\,d\xi$$

Since $\chi_{10}^*(w)$ represents a generalization of the Beber-Schafheitlin integral,

$$\chi_{10}^*(w) = \frac{2}{\pi\sqrt{[(r+u)^2-w^2]\,[w^2-(r-u)^2]}} \qquad (|r-u| < w < r+u)$$

For the evaluation of $\psi_{10}^*(w)$ at $z = 0$, we take note of the relation

$$J_0(w\xi) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos(w\xi\sin\theta)\,d\theta$$

so that

$$\psi_{10}^*(w) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} d\theta \int_0^\infty e^{\xi z}\cos(w\xi\sin\theta)\,\frac{\sin\sqrt{g\xi}\,(t-\tau)}{\sqrt{g\xi}}\,d\xi \tag{3.5}$$

Further evaluations will be carried out for $z = 0$. The correctness of the interchange of the order of integration in the formulas below is easily justified [7]. We carry out this interchange in (3.5) with the aid of the relation

$$w\xi\sin\theta = v^2$$

We also introduce the parameter $\beta = \frac{t-\tau}{2}\left(\frac{g}{w\sin\theta}\right)^{1/2}$. Now,

$$\psi_{10}^*(w)\,|_{z=0} = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{2d\theta}{\sqrt{gw\sin\theta}}\int_0^\infty \cos v^2 \sin 2\beta v\,dv$$

Elementary manipulations yield

$$E(\beta) \equiv \int_0^\infty \cos v^2 \sin 2\beta v\,dv = \beta\int_0^1 \sin\beta^2(1-\lambda^2)\,d\lambda$$

In this manner,

$$\psi_{10}^*(w)|_{z=0} = \frac{2}{\pi}\int\limits_0^{1/2\pi} \frac{2E(\beta)\,d\theta}{\sqrt{gw\sin\theta}} = \frac{2(t-\tau)}{\pi w}\int\limits_0^{1} d\lambda \int\limits_0^{1/2\pi} \sin\frac{\mu}{\sin\theta}\,\frac{d\theta}{\sin\theta}$$

$$\left(\mu = \frac{g(t-\tau)^2}{4w}(1-\lambda^2)\right)$$

Let us introduce an ew variable of integration ν into the second integral of the last formula

$$\frac{1}{\sin\theta} = \operatorname{ch}\nu, \qquad \frac{d\theta}{\sin\theta} = -d\nu$$

Then

$$\psi_{10}^*(w)|_{z=0} = \frac{t-\tau}{w}\int\limits_0^1 J_0\left[\frac{g(t-\tau)^2}{4w}(1-\lambda^2)\right]d\lambda$$

Once more we change the variable of integration

$$1-\lambda^2 = \cos\vartheta, \qquad d\lambda = \frac{1}{\sqrt{2}}\cos\frac{\vartheta}{2}\,d\vartheta$$

and we obtain

$$\psi_{10}^*(w)|_{z=0} = \frac{t-\tau}{w\sqrt{2}}\int\limits_0^{1/2\pi} J_0\left[\frac{g(t-\tau)^2}{4w}\cos\vartheta\right]\cos\frac{\vartheta}{2}\,d\vartheta =$$

$$= \frac{\pi(t-\tau)}{2\sqrt{2}w} J_{\frac{1}{4}}\left[\frac{g(t-\tau)^2}{8w}\right] J_{-\frac{1}{4}}\left[\frac{g(t-\tau)^2}{8w}\right] \tag{3.6}$$

Finally, writing the desired expression as

$$L|_{z=0} = \frac{t-\tau}{\sqrt{2}}\int\limits_{|r-u|}^{r+u} J_{\frac{1}{4}}\left[\frac{g(t-\tau)^2}{8w}\right] J_{-\frac{1}{4}}\left[\frac{g(t-\tau)^2}{8w}\right] \times$$

$$\times \frac{dw}{\sqrt{[(r+u)^2-w^2][w^2-(r-u)^2]}} = \frac{t-\tau}{\sqrt{2}} M(r, u, t, \tau) \tag{3.7}$$

it is not difficult to find the limiting value for S_m at the surface, for $m = 1, \ldots, n$:

$$S_m|_{z=0} = -\frac{t-\tau}{\sqrt{2}\,2^{m-1}(m-1)!\,r_0^{m+1}}\int\limits_0^{r_0} \frac{\partial M(r, u, t, \tau)}{\partial u}(r_0^2-u^2)^{m-1}u^2\,du \tag{3.8}$$

For the case $m = 0$, we obviously have

$$S_0 = -\frac{\partial}{\partial r_0}[L|_{u=r_0}]$$

Therefore,

$$S_0\,|_{z=0} = -\frac{t-\tau}{\sqrt{2}}\,\frac{\partial M(r,\, r_0,\, t,\, \tau)}{\partial r_0} \tag{3.9}$$

As a consequence, we obtain for the expressions (3.1)

$$\zeta_{n0}(r,\, t) = -\frac{1}{\sqrt{2}}\int_0^t \lambda_n(\tau)\, r_0^{2n+1}(\tau)\,\frac{\partial M(r,\, r_0,\, t,\, \tau)}{\partial r_0}(t-\tau)\,d\tau$$

$$\zeta_{nm}(r,\, t) = -\frac{1}{\sqrt{2}\,2^{m-1}(m-1)!}\int_0^t \lambda_n(\tau)\, r_0^{2n-2m}(\tau)\,(t-\tau)\,d\tau\times \tag{3.10}$$

$$\times\int_0^{r_0}\frac{\partial M(r,\, u,\, t,\, \tau)}{\partial u}(r_0^2-u^2)^{m-1}u^2du \qquad (m=1,\ldots,n)$$

The shape of the free surface is then given by

$$\zeta(r,\, t) = \frac{1}{\rho\sqrt{2}}\sum_{n=0}^{\infty}\int_0^t \lambda_n(\tau)\, r_0^{2n+1}(\tau)\,\frac{\partial M(r,\, r_0,\, t,\, \tau)}{\partial r_0}(t-\tau)\,dt +$$

$$+\frac{1}{\rho\sqrt{2}}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(-1)^m n!2}{(n-m)!\,(m-1)!}\int_0^t \lambda_n(\tau)\, r_0^{2n-2m}(\tau)\,(t-\tau)\,d\tau\times$$

$$\times\int_0^{r_0}\frac{\partial M(r,\, u,\, t,\, \tau)}{\partial u}(r_0^2-u^2)^{m-1}u^2du$$

It can be shown that this formula can be reduced to

$$\zeta(r,\, t) = \frac{1}{\rho\sqrt{2}}\int_0^t \frac{\partial M(r,\, r_0,\, t,\, \tau)}{\partial r_0}\, p_0[r_0(\tau),\, \tau]\, r_0(\tau)\,(t-\tau)\,d\tau -$$

$$-\frac{1}{\rho\sqrt{2}}\int_0^t (t-\tau)\,d\tau\int_0^{r_0}\frac{\partial M(r,\, u,\, t,\, \tau)}{\partial u}\,\frac{\partial p_0(u,\, \tau)}{\partial u}\,udu \tag{3.11}$$

where p_0 is the function given in a series form in (2.3).

4. Evaluation of the function M. The function $M(r,\, r_0,\, t,\, \tau)$, defined through (3.7), can be represented as follows:

$$M(r,\, r_0,\, t,\, \tau) = \frac{M_0(A,\, k)}{r+r_0} \qquad \left(A = \frac{g\,(t-\tau)^2}{8\,(r+r_0)},\quad k = \frac{|r-r_0|}{r+r_0}\right) \tag{4.1}$$

Here

$$M_0(A,\, k) = \int_k^1 J_{\frac{1}{4}}\left(\frac{A}{x}\right) J_{-\frac{1}{4}}\left(\frac{A}{x}\right)\frac{dx}{\sqrt{(1-x^2)\,(x^2-k^2)}}$$

Then,

$$\frac{\partial M}{\partial r_0} = -\frac{1}{(r+r_0)^2}\begin{cases}\Phi(A,k)+k(1+k)\,[\Psi(A,k)-L(A,k)] & \text{for } r>r_0\\ \Phi(A,k)-k(1-k)\,[\Psi(A,k)-L(A,k)] & \text{for } r<r_0\end{cases} \qquad (4.2)$$

Here

$$\Phi(A,k)=\int_k^1\left[J_{\frac{1}{4}}\left(\frac{A}{x}\right)J_{-\frac{1}{4}}\left(\frac{A}{x}\right)\frac{A}{x}\right]'\frac{dx}{\sqrt{(1-x^2)(x^2-k^2)}}$$

$$\Psi(A,k)=\int_k^1\left[J_{\frac{1}{4}}\left(\frac{A}{x}\right)J_{-\frac{1}{4}}\left(\frac{A}{x}\right)-J_{\frac{1}{4}}\left(\frac{A}{k}\right)J_{-\frac{1}{4}}\left(\frac{A}{k}\right)\right]\frac{dx}{(x^2-k^2)\sqrt{(1-x^2)(x^2-k^2)}}$$

$$L(A,k)=J_{\frac{1}{4}}\left(\frac{A}{k}\right)J_{-\frac{1}{4}}\left(\frac{A}{k}\right)\frac{E(\sqrt{1-k^2})-k^2K(\sqrt{1-k^2})}{k^2(1-k^2)}$$

where K and E are the complete elliptic integrals of the first and second kind.

When the motion of the free surface outside the region of the applied pressure is considered, then r always exceeds r_0, i.e. $k > k_0 \neq 0$. The integrals $\Phi(A, k)$ and $\Psi(A, k)$ can then be found by numerical integration as functions of parameters A and k; these functions do not depend on the actual form of p_0. However, the above formulas lead to excessive computational difficulties when the motion of the free surface within the region of applied pressure is considered. Since in that region the effect of the explosive pressures will tend to dominate, it is meaningful to study the problem of explosions above the surface of weightless incompressible fluids. Letting $g = 0$ in Equation (4.1), we obtain

$$M=\frac{2\sqrt{2}}{\pi}\,\frac{K(\sqrt{1-k^2})}{r+r_0} \qquad (4.3)$$

Hence

$$\frac{\partial M}{\partial r_0}=\frac{2\sqrt{2}}{\pi(r^2-r_0^2)}-\frac{2\sqrt{2}}{\pi(r+r_0)^2}\begin{cases}\dfrac{kK(\sqrt{1-k^2})-E(\sqrt{1-k^2})+(1-k)}{k(1-k)} & \text{for } r>r_0\\ \dfrac{kK(\sqrt{1-k^2})+E(\sqrt{1-k^2})-(1+k)}{k(1+k)} & \text{for } r<r_0.\end{cases} \qquad (4.4)$$

These formulas make it possible to determine the displacement of the free surface for arbitrary values of r. For instance, we find $\zeta(0, t)$ of a weightless fluid when the applied pressure is independent of r, $p_0 = F(r)$. It is easy to see that

$$\zeta(0,t)=-\frac{1}{\rho}\int_0^t\frac{F(\tau)(t-\tau)\,d\tau}{r_0(\tau)}$$

If H stands for the height above the liquid surface at which the explosion occurs, t_H for the time between the explosion and the first arrival of the shock at the free surface, and if $R_0(t_H+\tau)$ designates the spread of the shock in the gas above the liquid, then

$$r_0(\tau)=\sqrt{R_0^2(t_H+\tau)-H^2}$$

For small values* of τ

$$r_0(\tau)=\tau^{1/2}\sqrt{2HR_0'(t_H)+O(\tau)}$$

It follows that the displacement of the free liquid surface directly under the center of the explosion is always finite for arbitrary propagation rates of the shock in the gaseous medium above the liquid, provided $F(\tau)$ satisfies suitable conditions.

Incidentally, let us remark that in the case of explosions above ponderable fluids, it is possible to take advantage of the smallness of the ratio A/k for the evaluation of the free-surface displacements outside the pressure-affected region even when k is small. Developing the product of the Bessel functions, occuring in (4.1) in the integral for $M_0(A,\ k)$, into a series and keeping only its first two terms, we find the expression for $M(r,\ r_0,\ t,\ \tau)$

$$M(r,r_0,t,\tau)\approx\frac{2\sqrt{2}}{\pi(r+r_0)}\left[K(\sqrt{1-k^2})-\frac{8}{15}\left(\frac{A}{k}\right)^2E(\sqrt{1-k^2})\right] \quad (4.5)$$

Partial differentiation of $M(r,\ r_0,\ t,\ \tau)$ with respect to the parameter r_0 provides the remaining information necessary for the evaluation of the surface displacements.

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* Undoubtedly, the prime in R_0' represents the derivative of R_0 and $O(\tau)$, terms of order τ or smaller.

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